ABSTRACT ALGEBRA
TOPIC 3: COMPLEX NUMBERS

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1. Complex Algebra

The set of *complex numbers* is

$$\mathbb{C} = \{a + ib \mid b \in \mathbb{R}, i^2 = -1\}.$$

Let $z_1, z_2 \in \mathbb{C}$. Then $z_1 = x_1 + iy_1$ and $z_2 = x_2 iy_2$ for some $x_1, y_1, x_2, y_2 \in \mathbb{R}$. Define addition and multiplication in $\mathbb{C}$ by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2);$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2).$$

Thus to add or multiply complex numbers, treat i like a variable, add or multiply, replace i^2 with -1, and combine like terms.

One can show that these operations have the following properties:

(F1) $a + b = b + a$ for every $a, b \in \mathbb{C}$;
(F2) $(a + b) + c = a + (b + c)$ for every $a, b, c \in \mathbb{C}$;
(F3) there exists $0 \in \mathbb{C}$ such that $a + 0 = a$ for every $a \in \mathbb{C}$;
(F4) for every $a \in \mathbb{C}$ there exists $b \in \mathbb{C}$ such that $a + b = 0$;
(F5) $ab = ba$ for every $a, b \in \mathbb{C}$;
(F6) $(ab)c = a(bc)$ for every $a, b, c \in \mathbb{C}$;
(F7) there exists $1 \in \mathbb{C}$ such that $a \cdot 1 = a$ for every $a \in \mathbb{C}$;
(F8) for every $a \in \mathbb{C} \smallsetminus \{0\}$ there exists $c \in \mathbb{C}$ such that $ac = 1$;
(F9) $a(b + c) = ab + ac$ for every $a, b, c \in \mathbb{C}$.

Together, these properties state that $\mathbb{C}$ is a *field*. Note that

- $0 = 0 + i0$;
- $1 = 1 + i0$;
- $-(x + iy) = -x + i(-y) = -x - iy$;
- $(x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}$.

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2. Complex Geometry

Let $z = x + iy$ be an arbitrary complex number. The *real part* of z is $\Re(z) = x$. The *imaginary part* of z is $\Im(z) = y$. We view $\mathbb{R}$ as the subset of $\mathbb{C}$ consisting of those elements whose imaginary part is zero.

We graph complex number on the xy-plane, using the real part as the first coordinate and the imaginary part as the second coordinate. Under this interpretation, the set $\mathbb{C}$ becomes a real vector space of dimension two, with scalar multiplication given by complex multiplication by a real number. We call this vector space the *complex plane*.

Thus the geometric interpretation of complex addition is vector addition.

Let $z = x + iy$ be an arbitrary complex number. The *conjugate* of z is

$$\overline{z} = x - iy.$$

This is the mirror image of z under reflection across the real axis. Note that

$$z + \overline{z} = (x + iy) + (x - iy) = 2x = 2\Re(z).$$

The *modulus* of z is

$$|z| = \sqrt{x^2 + y^2}.$$

This is the length of z as a vector. Note that

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

The *angle* of z, denoted by $\angle(z)$, is the angle between the vectors $(1, 0)$ and (x, y) in the real plane $\mathbb{R}^2$; this is well-defined up to a multiple of 2π.

Let $r = |z|$ and $\theta = \angle(z)$. Then $x = r\cos\theta$ and $y = r\sin\theta$. Define a function

$$\text{cis} : \mathbb{R} \to \mathbb{C} \quad \text{by} \quad \text{cis}(\theta) = \cos\theta + i\sin\theta.$$

Then $z = r\,\text{cis}(\theta)$; this is the *polar representation* of z.

Recall the trigonometric formulae for the cosine and sine of the sum of angles:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \quad \text{and} \quad \sin(A+B) = \cos A\sin B + \sin A\cos B.$$

Let $z_1 = r_1\,\text{cis}(\theta_1)$ and $z_2 = r_2\,\text{cis}(\theta_2)$. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\ &= r_1 r_2\,\text{cis}(\theta_1 + \theta_2). \end{aligned}$$

Thus the geometric interpretation of complex multiplication is:

(a) The radius of the product is the product of the radii;
(b) The angle of the product is the sum of the angles.

Example 1. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = 2z$. Then f dilates the complex plane by a factor of 2.

Example 2. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = iz$. Then f rotates the complex plane by 90 degrees.

Example 3. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = (1 + i)z$. Note that $|1 + i| = \sqrt{2}$ and $\angle(1 + i) = \frac{\pi}{4}$. Then f dilates the complex plane by a factors of $\sqrt{2}$ and rotates it by 45 degrees.

3. Complex Powers and Roots

A special case of complex multiplication is exponentiation by a natural number; a simple proof by induction shows that

Theorem 1. (DeMoivre's Theorem)
Let $\theta \in \mathbb{R}$. Then

$$(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta).$$

Let $z = r\operatorname{cis}(\theta)$ and let $n \in \mathbb{N}$. Then $z^n = r^n \operatorname{cis}(n\theta)$.

The *unit circle* in the complex plane is

$$\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Note that if $u_1, u_2 \in \mathbb{U}$, then $u_1 u_2 \in \mathbb{U}$.

Let $\zeta \in \mathbb{C}$ and suppose that $\zeta^n = 1$. We call ζ an n^{th} *root of unity*. Note that if $\zeta = \operatorname{cis}(2\pi/n)$, then

$$\zeta^n = \operatorname{cis}^n(2\pi/n) = \operatorname{cis}(2\pi n/n) = \operatorname{cis}(2\pi) = 1,$$

so n^{th} roots of unity always exist. In fact, for $k \in \mathbb{Z}$, $\zeta^k = \operatorname{cis}(2\pi k/n)$ is also an $(^{\text{th}}n)$ root of unity, since

$$(\zeta^k)^n = (\zeta^n)^k = 1^k = 1.$$

Moreover, $\zeta^i = \zeta^j$ if and only if $i \equiv j \pmod n$, in particular, $\zeta^n = \zeta^0 = 1$. Thus there are exactly n distinct complex numbers which are n^{th} roots of unity; they form the set

$$\mathbb{U}_n = \{\zeta^k \mid \zeta = \operatorname{cis}(2\pi/n), k = 0, 1, \dots, n-1\}.$$

If $\alpha \in \mathbb{U}_n$, we call α a *primitive* n^{th} *root of unity* if $\alpha^j \neq 1$ for $j = 1, \dots, n-1$. If α is a primitive n^{th} root of unity, then $\mathbb{U}_n = \{\alpha^k \mid k = 0, \dots, n-1\}$.

If one graphs the n^{th} roots of unity in the complex plane, the points lie on the unit circle and they are the vertices of a regular n-gon, with one vertex always at the point $1 = 1 + i0$.

Let $z = r\operatorname{cis}(\theta)$. Then z has exactly n distinct n^{th} roots; they are

$$\sqrt[n]{z} = \zeta^m \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n}\right), \quad \text{where} \quad \zeta = \operatorname{cis}\left(\frac{2\pi}{n}\right) \text{ and } m \in \{0, \dots, n-1\}.$$

The algebraic importance of the complex numbers, and the original motivation for their study, is exemplified by the next theorem. This was first conjectured in the 1500's, but was not proven until the doctoral dissertation of Carl Friedrich Gauss in 1799 at the age of 22. Incidentally, was the first to prove the constructibility of a regular 17-gon, at an even earlier age.

Theorem 2. (The Fundamental Theorem of Algebra)
Every polynomial with complex coefficients has a zero in $\mathbb{C}$.

From this, it follows that every polynomial with complex coefficients factors completely into the product of linear polynomials with complex coefficients.

4. Complex Analysis

The *distance* between complex number z_1 and z_2 is $|z_1 - z_2|$. This is standard distance in the complex plane, and allows us to precisely define what it means for a complex function to be continuous or differentiable. Moreover, the theory of sequences and series carries over to the complex numbers.

We use the power series expansion of various familiar real-valued functions to motivate the definitions of their complex analogs. In each case, one may use the ratio test to see that the radius of convergence is infinite, so the functions are defined on the entire complex plane.

Define the complex exponential function

$$\exp : \mathbb{C} \to \mathbb{C} \quad \text{by} \quad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Define the complex sine function by

$$\sin : \mathbb{C} \to \mathbb{C} \quad \text{by} \quad \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

Define the complex cosine function by

$$\cos : \mathbb{C} \to \mathbb{C} \quad \text{by} \quad \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

Note that exp, sin, and cos, when restricted to $\mathbb{R} \subset \mathbb{C}$, are defined so as to be consistent with other definitions of these real functions. In particular, we still have $e = \exp(1)$.

Define $\log : D \to \mathbb{C}$ to be an inverse function of exp, where

$$D = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x < 0\};$$

then log is continuous on D. Note that $\log(1) = e$.

Let $w, z \in \mathbb{C}$. We define w^z by

$$w^z = \exp(z \log(w)).$$

Thus $\exp(z) = e^z$.

Compute that

$$\exp(iz) = \cos(z) + i \sin(z).$$

In particular, if z is the complex number $i\theta$, where $\theta \in \mathbb{R}$, we have

Theorem 3. (Euler's Theorem) *Let $\theta \in \mathbb{R}$. Then*

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

5. Exercises

The *rectangular form* of a complex number is $z = a + bi$.
The *polar form* of a complex number is $z = r \operatorname{cis} \theta$.

Exercise 1. Let $z = 7 - 2i$ and $w = 5 + 3i$.
Compute the following, expressed in rectangular form.

(a) $z + w$
(b) $3z - 8w$
(c) zw
(d) $\frac{z}{w}$
(e) $\overline{z}$ and $|z|$

Exercise 2. Find the rectangular and polar forms of all sixth roots of unity.

Exercise 3. Find the rectangular and polar forms of all solutions to the equation $z^6 - 8 = 0$.

Exercise 4. Find the rectangular and polar forms of all solutions to the equation $z^6 - a = 0$, where $a = \sqrt{3} + i$.

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